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On the Antipode of a Quasitriangular Hopf Algebra

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Let (A, R) be a quasitriangular Hopf algebra with antipode s in the category of vector spaces over a field k. In this paper we show that s^2 is inner, which means s is bijective, and when A is finite-dimensional there is a grouplike element $h \in A$ such that $s^4(a) = hah^{-1}$ for all $a \in A$.

Suppose that A is any Hopf algebra with antipode s over the field k. Proposition 1 of Section 1 gives a sufficient condition for s^2 to be inner which we apply to quasitriangular Hopf algebras in Section 2. Its statement and proof are based on calculations found in [2, pp. 66–67].

Generally s^2 need not be inner. Suppose that A is finite-dimensional. If s^2 were inner, then all ideals of A would be invariant under s^2, and hence all subcoalgebras of the dual Hopf algebra A^* would be invariant under S^2, where $S = s^*$ is the antipode of A^*. There are examples [5] over algebraically closed fields in which the latter is not the case.

Section 2 begins with a discussion of properties of quasitriangular Hopf algebras (A, R) over the field k used in this paper. Write $R = \sum R^{(1)} \otimes R^{(2)} \in A \otimes A$ and set $u = \sum s(R^{(2)}) R^{(1)}$. The main result of Section 2 is that u is invertible and $s^2(a) = uau^{-1}$ for all $a \in A$. This result was established in [2, Exercise 7.3.6] under the hypothesis that s is bijective.

If A is any finite-dimensional Hopf algebra with antipode s over the field k, there are distinguished grouplike elements $g \in A$ and $\alpha \in A^*$ which relate left and right integrals and which together describe s^4. In Section 3 we consider s^4 when (A, R) is a finite-dimensional quasitriangular Hopf algebra. Let $h = vu$, where $u = \sum s(R^{(2)}) R^{(1)}$ is defined as above and $v = s(u)^{-1}$. By relating h to g and α, we are able to show that h is a grouplike element and that $s^4(a) = hah^{-1}$ for all $a \in A$. When A is unimodular we prove that $h = g$. Thus the product vu does not depend on R in the unimodular case.

We assume that the reader has some familiarity with the elementary aspects of the theory of Hopf algebras. A recommended reference is [6].

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1. A Sufficient Condition for s^2 to be Inner

Suppose that A is a Hopf algebra with antipode s over the field k. Let $\Delta^{cop}(a)=\sum a_{(2)}\otimes a_{(1)}$ for $a\in A$ denote the coproduct on A derived by "twisting" comultiplication, and let A^{cop} denote the resulting bialgebra. Define a right module action of A on itself by

$$b\lhd a=\sum s(a_{(2)})\, ba_{(1)} \qquad \text{for} \quad a\in A.$$

Observe that

$$\begin{aligned} s^2(a)\, b &= \sum s^2(a_{(1)})\, b(\varepsilon(a_{(2)})\, 1) \\ &= \sum s^2(a_{(1)})\, bs(a_{(2)})\, a_{(3)} \\ &= \sum s((s(a_{(1)}))_{(2)})\, b(s(a_{(1)}))_{(1)}\, a_{(2)} \\ &= \sum (b\lhd s(a_{(1)}))\, a_{(2)} \end{aligned}$$

for all a, $b\in A$ since s is a coalgebra antihomomorphism [6, Proposition 4.0.1]. Thus left and right multiplication in A are related by

$$s^2(a)\, b=\sum (b\lhd s(a_{(1)}))\, a_{(2)} \qquad \text{for all} \quad a, b\in A. \tag{1}$$

Proposition 1. *Suppose that A is a Hopf algebra with antipode s over the field k. Suppose further that $R=\sum R^{(1)}\otimes R^{(2)}\in s(A)\otimes A$ has an inverse in $A\otimes A$, and that $\Delta^{cop}(a)=R(\Delta(a))\, R^{-1}$ for all $a\in A$. Then $u=\sum s(R^{(2)})\, R^{(1)}$ is invertible and $s^2(a)=uau^{-1}$ for all $a\in A$. In particular s is bijective.*

Proof. Let $a\in A$. We first consider implications of the hypothesis $\Delta^{cop}(a)=R(\Delta(a))\, R^{-1}$, which can be formulated $(\Delta^{cop}(a))\, R=R(\Delta(a))$.

Suppose that $X=\sum X^{(1)}\otimes X^{(2)}\in A\otimes A$ satisfies $(\Delta^{cop}(a))\, X=X(\Delta(a))$ for all $a\in A$, which is the same as

$$\sum a_{(2)}X^{(1)}\otimes a_{(1)}X^{(2)}=\sum X^{(1)}a_{(1)}\otimes X^{(2)}a_{(2)} \qquad \text{for all} \quad a\in A. \tag{2}$$

Let $x=\sum s(X^{(2)})\, X^{(1)}$. Noting that s is an algebra antihomomorphism [6, Proposition 4.0.1], we apply $I\otimes s$ to both sides of Eq. (2), exchange tensors, and then multiply to obtain

$$x\lhd a=\varepsilon(a)\, x \qquad \text{for all} \quad a\in A. \tag{3}$$

By Eqs. (1) and (3)

$$s^2(a)\, x=xa \qquad \text{for all} \quad a\in A. \tag{4}$$

Write $R^{-1}=\sum U^{(1)}\otimes U^{(2)}$. Since $R^{-1}(\Delta^{cop}(a))=(\Delta(a))\,R^{-1}$ for all $a\in A$, we have that $\sum U^{(1)}a_{(2)}\otimes U^{(2)}a_{(1)}=\sum a_{(1)}U^{(1)}\otimes a_{(2)}U^{(2)}$, and hence $\sum a_{(2)}U^{(2)}\otimes a_{(1)}U^{(1)}=\sum U^{(2)}a_{(1)}\otimes U^{(1)}a_{(2)}$ for all $a\in A$. Set $v=\sum s(U^{(1)})\,U^{(2)}$. Then by Eqs. (2) and (4) we conclude that

$$s^2(a)\,u=ua \qquad \text{and} \qquad s^2(a)\,v=va \qquad \text{for all} \quad a\in A. \tag{5}$$

Now $\sum U^{(1)}R^{(1)}\otimes U^{(2)}R^{(2)}=1\otimes 1$. Applying $s\otimes I$ to this equation and multiplying, we obtain $\sum s(R^{(1)})\,s(U^{(1)})\,U^{(2)}R^{(2)}=1$, or $\sum s(R^{(1)})\,vR^{(2)}=1$. Therefore by Eq. (5) we have that $\sum s(R^{(1)})\,s^2(R^{(2)})\,v=1$, or equivalently

$$s(u)\,v=1. \tag{6}$$

By Eq. (6) it follows that v has a left inverse, and hence s is injective by Eq. (5).

Now we use the hypothesis $R\in s(A)\otimes A$ to show that v has a right inverse. Under this hypothesis $u=\sum s(R^{(2)})\;R^{(1)}\in s(A)$, so $u=s(w)$ for some $w\in A$. Thus by Eqs. (6) and (5) it follows that $1=s(u)\,v=s^2(w)\,v=vw$. Therefore v is invertible, and hence $s^2(a)=vav^{-1}$ for all $a\in A$ by Eq. (5). In particular s is bijective. This implies that u is invertible also, and hence $s^2(a)=uau^{-1}$ for all $a\in A$ by Eq. (5) again. The proof of the proposition is complete.

An easy corollary to the proof is that when A is finite-dimensional, and R, $U\in A\otimes A$ satisfy $\Delta^{cop}(a)=R(\Delta(a))\,U$ for all $a\in A$, then s^2 is inner. The last proposition of Section 3 is an improvement of this observation.

2. Quasitriangular Hopf Algebras

In this section we show that the hypothesis of Proposition 1 is satisfied for a quasitriangular Hopf algebra (A, R) with antipode s in the category of vector spaces over the field k, and thus it will follow that s^2 is inner. We begin by giving a definition of quasitriangular Hopf algebras which is equivalent to the one of [1, p. 811] in our context.

Definition. A *quasitriangular Hopf algebra* in the category of vector spaces over the field k is a pair (A, R), where A is a Hopf algebra over k, and $R=\sum R^{(1)}\otimes R^{(2)}\in A\otimes A$ satisfies

(QT.1) $\sum \Delta R^{(1)}\otimes R^{(2)}=\sum R^{(1)}\otimes r^{(1)}\otimes R^{(2)}r^{(2)}$, where $r=R$,

(QT.2) $\sum \varepsilon(R^{(1)})\,R^{(2)}=1$,

(QT.3) $\sum R^{(1)}\otimes \Delta R^{(2)}=\sum R^{(1)}r^{(1)}\otimes r^{(2)}\otimes R^{(2)}$, where $r=R$,

(QT.4) $\sum R^{(1)}\,\varepsilon(R^{(2)})=1$, and

(QT.5) $(\Delta^{cop}(a))\,R=R(\Delta(a))$ for all $a\in A$.

Let A be a Hopf algebra over the field k, and let $F: A \otimes A \to \mathrm{Hom}_k(A^*, A)$ be the (injective) linear map defined by $F(a \otimes b)(p) = p(a)\, b$ for all a, $b \in A$ and $p \in A^*$. Now suppose that $R = \sum R^{(1)} \otimes R^{(2)} \in A \otimes A$ and set $f = F(R)$.

It is easy to see that R satisfies (QT.1) and (QT.2) if and only if f is an algebra map. If f is an algebra map then R is invertible, for:

LEMMA 1. *Suppose that A is a Hopf algebra with antipode s over the field k, and $R \in A \otimes A$ satisfies* (QT.1) *and* (QT.2). *Then R is invertible with inverse $R^{-1} = \sum s(R^{(1)}) \otimes R^{(2)}$.*

Proof. First observe that (QT.2) implies that $1 \otimes 1 = \sum \varepsilon(R^{(1)})\, 1 \otimes R^{(2)}$. By definition of the antipode $\sum s(a_{(1)})\, a_{(2)} = \varepsilon(a)\, 1 = \sum a_{(1)} s(a_{(2)})$ for all $a \in A$. Now using (QT.1) it is easy to see that $1 \otimes 1 = \sum s(R^{(1)})\, r^{(1)} \otimes R^{(2)} r^{(2)} = \sum R^{(1)} s(r^{(1)}) \otimes R^{(2)} r^{(2)}$. This concludes the proof of the lemma.

Suppose that R is invertible and that (QT.1) is satisfied. Then $\sum \varepsilon(R^{(1)})\, R^{(2)}$ is an invertible idempotent, so (QT.2) is satisfied. Likewise if R is invertible and (QT.3) is satisfied, then (QT.4) is satisfied also. Thus the definition of quasitriangular Hopf algebras given above agrees with the one given in [1, p. 811] specialized to the category of vector spaces over k. From this point on we will refer to quasitriangular Hopf algebras in the category of vector spaces over k simply as quasitriangular Hopf algebras.

LEMMA 2. *Suppose that A is a Hopf algebra with antipode s over the field k, and $R \in A \otimes A$ satisfies* (QT.3) *and* (QT.4). *Then $\sum s(R^{(1)}) \otimes R^{(2)}$ is invertible with inverse $\sum s(R^{(1)}) \otimes s(R^{(2)})$. Thus $\sum R^{(1)} \otimes R^{(2)} = \sum s(R^{(1)}) \otimes s(R^{(2)})$ if (A, R) is quasitraingular.*

Proof. First observe that (QT.4) implies that $1 \otimes 1 = \sum R^{(1)} \otimes \varepsilon(R^{(2)})\, 1$. Therefore (QT.3) and (QT.4) imply that $1 \otimes 1 = \sum R^{(1)} r^{(1)} \otimes s(r^{(2)})\, R^{(2)} = \sum R^{(1)} r^{(1)} \otimes r^{(2)} s(R^{(2)})$. Since s is an algebra anti-homomorphism, applying $s \otimes I$ to these equations gives us that $\sum s(R^{(1)}) \otimes R^{(2)}$ is invertible with inverse $\sum s(R^{(1)}) \otimes s(R^{(2)})$. The last assertion now follows by Lemma 1. This completes the proof.

The two preceding lemmas are minor reformulations of material found in [2, p. 13].

Now suppose that A is finite-dimensional. Then it is easy to see that R satisfies (QT.3) and (QT.4) if and only if f is a coalgebra anti-homomorphism. In the finite-dimensional case F is an isomorphism of the algebra $A \otimes A$ and the convolution algebra $\mathrm{Hom}_k(A^*, A^{cop})$. The Hopf algebra A^* has antipode $S = s^*$. Since A is finite-dimensional s is bijective

[6, Corollary 5.1.6]. Since s is bijective A^{cop} is a Hopf algebra with antipode s^{-1}.

Suppose further that (A, R) is quasitriangular. Then the conclusions of the lemmas have an easy explanation in terms of the convolution algebra $\mathrm{Hom}_k(A^*, A^{cop})$. Since (A, R) is quasitriangular $f: A^* \to A^{cop}$ is a bialgebra homomorphism. Since f is an algebra homomorphism, f has an inverse in the convolution algebra which is $f \circ S$ by [6, Lemma 4.0.3.(1)]. Observe that $f \circ S$ corresponds to $\sum s(R^{(1)}) \otimes R^{(2)}$. Since f is a coalgebra homomorphism, f has an inverse in the convolution algebra which is $s^{-1} \circ f$ by [6, Lemma 4.0.3.(2)]. Therefore $f \circ S = s^{-1} \circ f$, or $f = s \circ f \circ S$. It is easy to see that $s \circ f \circ S$ corresponds to $\sum s(R^{(1)}) \otimes s(R^{(2)})$.

Now suppose that (A, R) is any quasitriangular Hopf algebra with antipode s over the field k. By Lemma 2 it follows that $R \in s(A) \otimes s(A)$. As a consequence of Proposition 1 we now have:

THEOREM 1. *Suppose that (A, R) is a quasitriangular Hopf algebra with antipode s over the field k. Write $R = \sum R^{(1)} \otimes R^{(2)}$ and let $u = \sum s(R^{(2)}) R^{(1)}$. Then u is invertible, and $s^2(a) = uau^{-1}$ for all $a \in A$.*

It may be of interest to note that (QT.5) can be expressed in terms of A-module actions. Define A-module actions on A by

$$a \rightharpoonup b = ab \qquad \text{and} \qquad b \leftharpoonup a = ba$$

for a, $b \in A$, on A^* by

$$a \rightharpoonup p(b) = p(b \leftharpoonup a) \qquad \text{and} \qquad p \leftharpoonup a(b) = p(a \rightharpoonup b)$$

for a, $b \in A$ and $p \in A^*$, and on $\mathrm{Hom}_k(A^*, A)$ by

$$a \rightharpoonup f(p) = \sum a_{(1)} \rightharpoonup (f(p \leftharpoonup a_{(2)}))$$

and

$$f \leftharpoonup a(p) = \sum (f(a_{(1)} \rightharpoonup p)) \leftharpoonup a_{(2)}$$

for $a \in A$, $f \in \mathrm{Hom}_k(A^*, A)$, and $p \in A^*$. Notice that $a \rightharpoonup (f \leftharpoonup b) = (a \rightharpoonup f) \leftharpoonup b$ for all a, $b \in A$ and $f \in \mathrm{Hom}_k(A^*, A)$. Therefore

$$a \rightarrow f = \sum a_{(1)} \rightharpoonup f \leftharpoonup s(a_{(2)})$$

defines a left A-module action on $\mathrm{Hom}_k(A^*, A)$. It is easy to see that the A-module actions $\rightharpoonup$ and $\leftharpoonup$ on $\mathrm{Hom}_k(A^*, A)$ are related by

$$a \rightharpoonup f = \sum (a_{(1)} \rightarrow f) \leftharpoonup a_{(2)} \tag{7}$$

for all $a \in A$ and $f \in \mathrm{Hom}_k(A^*, A)$. The proof of the following proposition is left to the reader.

PROPOSITION 2. *Suppose that A is a Hopf algebra over the field k and $R = \sum R^{(1)} \otimes R^{(2)} \in A \otimes A$. Define $f \in \mathrm{Hom}(A^*, A)$ by $f(p) = \sum p(R^{(1)})\, R^{(2)}$ for $p \in A^*$. Give $\mathrm{Hom}_k(A^*, A)$ the A-module structures described above. Then the following are equivalent:*

(a) $\sum a_{(2)} R^{(1)} \otimes a_{(1)} R^{(2)} = \sum R^{(1)} a_{(1)} \otimes R^{(2)} a_{(2)}$ *for all* $a \in A$,

(b) $a \rightharpoonup f = f \leftharpoonup a$ *for all* $a \in A$,

(c) $a \rightharpoonup f = \varepsilon(a)\, f$ *for all* $a \in A$.

3. s^4 WHEN (A, R) IS FINITE-DIMENSIONAL

Suppose that (A, R) is a finite-dimensional quasitriangular Hopf algebra with antipode s over the field k. Write $R = \sum R^{(1)} \otimes R^{(2)}$, let $u = \sum s(R^{(2)})\, R^{(1)}$, and $v = s(u)^{-1}$. In this section we show that $h = vu$ is a grouplike element and $s^4(a) = hah^{-1}$ for all $a \in A$. Generally s^4 need not be inner in the finite-dimensional case [5]. The main results of this section are implications of equation (QT.5) specialized to $a = \Lambda$, where Λ is a non-zero left integral for A.

Suppose that A is any finite-dimensional Hopf algebra with antipode s over the field k. An element $\Lambda \in A$ is said to be a left (respectively right) integral if $a\Lambda = \varepsilon(a)\, \Lambda$ (respectively $\Lambda a = \varepsilon(a)\, \Lambda$) for all $a \in A$. Thus $\lambda \in A^*$ is a right (respectively left) integral if $\lambda p = p(1)\, \lambda$ (respectively $p\lambda = p(1)\, \lambda$) for all $p \in A^*$. The space of left (or right) integrals of A is a one-dimensional ideal. If $\Lambda \in A$ is a non-zero left (or right) integral, then A is a free A^*-module with basis Λ under the left action defined by $p \rightharpoonup a = \sum a_{(1)} p(a_{(2)})$ and also under the right action defined by $a \leftharpoonup p = \sum p(a_{(1)})\, a_{(2)}$ for $p \in A^*$ and $a \in A$. Thus if $\lambda \in A^*$ is a non-zero right (or left) integral, then A^* is a free A-module with basis λ under both A-module actions on A^* defined in the last section. We direct the reader to [6, Chap. 5] for a basic treatment of integrals with proofs.

A non-zero $a \in A$ is called a grouplike element if $\Delta(a) = a \otimes a$. If a is a grouplike element, then $\varepsilon(a) = 1$ and $s(a)$ is a grouplike element which is a multiplicative inverse of a. It is easy to see that the set of grouplike elements $G(A)$ of A is a group under multiplication, and that $G(A^*) = \mathrm{Alg}_k(A, k)$.

Let $\Lambda \in A$ be a non-zero left integral. Since Λ generates a one-dimensional ideal of A, there is a unique algebra homomorphism $\alpha \in \mathrm{Alg}_k(A, k) = G(A^*)$ such that $\Lambda a = \alpha(a)\, \Lambda$ for all $a \in A$. Now let $\lambda \in A^*$ be a non-zero right integral. Then there is a unique algebra homomorphism $g \in \mathrm{Alg}_k(A^*, k) = G(A^{**})$ such that $p\lambda = g(p)\,\lambda$ for all $p \in A^*$. We shall call $g \in A^{**} = A$ the distinguished grouplike element of A and α the distinguished grouplike element of A^*. By [3, Proposition 6]

$$s^4(a) = g(\alpha \rightharpoonup a \leftharpoonup \alpha^{-1})\,g^{-1} \qquad \text{for all} \quad a \in A. \tag{8}$$

By [4, Theorem 2(c)]

$$\sum A_{(2)} \otimes A_{(1)} = \sum A_{(1)} \otimes s^2(A_{(2)})\,g. \tag{9}$$

Let $\eta \in G(A^*) = \mathrm{Alg}_k(A, k)$. We say that $a \in A$ is a left (respectively right) η-integral if $ba = \eta(b)\,a$ (respectively $ab = \eta(b)\,a$) for all $b \in A$.

Lemma 3. *Suppose that A is a finite-dimensional Hopf algebra with antipode s over the field k, and let $\eta \in G(A^*)$. Then*

(a) *if $a \in A$ is a left η-integral, then $\sum a_{(1)} \otimes ba_{(2)} = \sum s(\eta \rightharpoonup b)\,a_{(1)} \otimes a_{(2)}$ for all $b \in A$,*

(b) *if $a \in A$ is a right η-integral, then $\sum a_{(1)}b \otimes a_{(2)} = \sum a_{(1)} \otimes a_{(2)}s(b \leftharpoonup \eta)$ for all $b \in A$.*

Proof. Let $a, b \in A$. The lemma follows directly from the equations $\sum a_{(1)} \otimes ba_{(2)} = \sum s(b_{(1)})(b_{(2)}a)_{(1)} \otimes (b_{(2)}a)_{(2)}$ and $\sum a_{(1)}b \otimes a_{(2)} = \sum (ab_{(1)})_{(1)} \otimes (ab_{(1)})_{(2)}\,s(b_{(2)})$. The derivation of the first equation is

$$\begin{aligned}\sum a_{(1)} \otimes ba_{(2)} &= \sum (\varepsilon(b_{(1)})\,1)\,a_{(1)} \otimes b_{(2)}a_{(2)} \\ &= \sum s(b_{(1)})\,b_{(2)}\,a_{(1)} \otimes b_{(3)}a_{(2)} \\ &= \sum s(b_{(1)})(b_{(2)}a)_{(1)} \otimes (b_{(2)}a)_{(2)}\end{aligned}$$

and the similiar derivation of the second is left to the reader. This concludes our proof.

Let (A, R) be a finite-dimensional quasitriangular Hopf algebra with antipode s over the field k. For $\eta \in G(A^*)$ define $g_\eta = \sum R^{(1)}\,\eta(R^{(2)})$, where $R = \sum R^{(1)} \otimes R^{(2)}$. There is an interesting relationship between the groups $G(A^*)$ and $G(A)$.

Proposition 3. *Suppose that (A, R) is a finite-dimensional quasitriangular Hopf algebra with antipode s over the field k, and let $\eta \in G(A^*)$. Then*

(a) $g_\eta \in G(A)$,

(b) *the map $G(A^*) \to G(A)$ given by $\eta \mapsto g_\eta$ is a group anti-homomorphism,*

(c) $(a \leftharpoonup \eta) g_\eta = g_\eta (\eta \rightharpoonup a)$ *for all* $a \in A$, *and*

(d) η *is central if and only if* g_η *is central.*

Proof. We first show part (a). By (QT.1) it follows that $\Delta g_\eta = \sum \Delta R^{(1)} \eta(R^{(2)}) = \sum R^{(1)} \otimes r^{(1)} \eta(R^{(2)} r^{(2)}) = \sum R^{(1)} \otimes r^{(1)} \eta(R^{(2)}) \eta(r^{(2)}) = g_\eta \otimes g_\eta$ since η is an algebra homomorphism. By (QT.2) we have that $\varepsilon(g_\eta) = \eta(\sum \varepsilon(R^{(1)}) R^{(2)}) = \eta(1) = 1$. Therefore $g_\eta \in G(A)$, and part (a) is established.

We next show part (b). For $\eta, \rho \in G(A^*)$ we see by (QT.3) that $g_{\eta\rho} = \sum R^{(1)} \eta\rho(R^{(2)}) = \sum R^{(1)} \eta(R^{(2)}_{(1)}) \rho(R^{(2)}_{(2)}) = \sum R^{(1)} r^{(1)} \eta(r^{(2)}) \rho(R^{(2)}) = g_\rho g_\eta$.

To prove part (c), recall that $\sum a_{(2)} R^{(1)} \otimes a_{(1)} R^{(2)} = \sum R^{(1)} a_{(1)} \otimes R^{(2)} a_{(2)}$ for all $a \in A$ by (QT.5). Applying $I \otimes \eta$ to both sides of this equation gives part (c).

We finally prove part (d). First observe that $p(a \leftharpoonup q) = qp(a)$ and $p(q \rightharpoonup a) = pq(a)$ for all $p, q \in A^*$ and $a \in A$. Therefore $q \in A^*$ is central if and only if $a \leftharpoonup q = q \rightharpoonup a$ for all $a \in A$. Since η is invertible, it follows that $A \leftharpoonup \eta = A = \eta \rightharpoonup A$. Thus part (d) follows by part (c). The proof of the proposition is complete.

Part (c) of Proposition 3 has an interesting interpretation. Let $\eta \in G(A^*)$. Then $T_\eta : A^* \to A^*$ defined by $T_\eta(p) = \eta p \eta^{-1}$ for $p \in A^*$ is an automorphism of Hopf algebras. Observe that $T_\eta = t_\eta^*$ where $t_\eta : A \to A$ is defined by $t_\eta(a) = \eta^{-1} \rightharpoonup a \leftharpoonup \eta$ for $a \in A$. By part (c) of Proposition 3 we have that $t_\eta(a) = g_\eta a g_\eta^{-1}$ for $a \in A$. In particular t_η is itself inner.

If (A, R) is a finite-dimensional quasitriangular Hopf algebra with antipode s over the field k, then by the comments made in the preceding paragraph and Eq. (8) it follows that $s^4(a) = hah^{-1}$ for all $a \in A$, where $h = g g_{\alpha^{-1}}$. The grouplike element h can be computed directly from R.

THEOREM 2.[1] *Suppose that* (A, R) *is a finite-dimensional quasitriangular Hopf algebra over the field k. Write* $R = \sum R^{(1)} \otimes R^{(2)}$, *let* $u = \sum s(R^{(2)}) R^{(1)}$, *and* $v = s(u)^{-1}$. *Suppose that g is the distinguished grouplike element of A, that* α *is the distinguished grouplike element of* A^*, *and let* $h = g g_{\alpha^{-1}}$. *Then*

(a) $g = v(\sum s(R^{(2)} \leftharpoonup \alpha) R^{(1)})$,

(b) $g = v u g_\alpha$, *and*

(c) $h = vu$. *Thus vu is a grouplike element, and* $s^4(a) = (vu)\, a (vu)^{-1}$ *for all* $a \in A$.

[1] Part (c) and Corollary 2 below are derived in Section 3 of [DRIN]. Part (b) is Proposition 6.1 of [DRIN] for A^{op}, and is proved by different methods there. The author was made aware of [DRIN] after this paper was accepted.

Proof. We first show part (a). Let $\Lambda \in A$ be a non-zero left integral. Then Λ is a left ε-integral and a right α-integral. Since $\sum \Lambda_{(2)}R^{(1)} \otimes \Lambda_{(1)}R^{(2)} = \sum R^{(1)}\Lambda_{(1)} \otimes R^{(2)}\Lambda_{(2)}$ by (QT.5), it follows by Lemma 3 that

$$\sum \Lambda_{(2)}s(R^{(2)} \leftharpoonup \alpha)\, R^{(1)} \otimes \Lambda_{(1)} = \sum R^{(1)}s(R^{(2)})\, \Lambda_{(1)} \otimes \Lambda_{(2)}.$$

By Eq. (9) we have that $\sum \Lambda_{(1)} \otimes \Lambda_{(2)} = \sum s^2(\Lambda_{(2)})\, g \otimes \Lambda_{(1)}$. Therefore

$$\sum \Lambda_{(2)}s(R^{(2)} \leftharpoonup \alpha)\, R^{(1)} \otimes \Lambda_{(1)} = \sum R^{(1)}s(R^{(2)})\, s^2(\Lambda_{(2)})\, g \otimes \Lambda_{(1)}.$$

Since $(A, \leftharpoonup)$ is a free right A^*-module with basis Λ, there is some $p \in A^*$ such that $\Lambda \leftharpoonup p = 1$. Thus applying $I \otimes p$ to both sides of the equation above we have that $\sum s(R^{(2)} \leftharpoonup \alpha)\, R^{(1)} = \sum R^{(1)}s(R^{(2)})\, g$. Now $\sum R^{(1)}s(R^{(2)}) = \sum s(R^{(1)})\, s^2(R^{(2)}) = s(u) = v^{-1}$ by Lemma 2. Therefore $\sum s(R^{(2)} \leftharpoonup \alpha)\, R^{(1)} = v^{-1}g$, and part (a) follows.

To establish part (b), we show that $\sum s(R^{(2)} \leftharpoonup \alpha)\, R^{(1)} = ug_\alpha$. Now $\sum R^{(1)} \otimes \Delta R^{(2)} = \sum R^{(1)}r^{(1)} \otimes r^{(2)} \otimes R^{(2)}$ by (QT.3). Therefore $\sum R^{(1)} \otimes R^{(2)} \leftharpoonup \alpha = \sum R^{(1)}r^{(1)}\alpha(r^{(2)}) \otimes R^{(2)} = \sum R^{(1)}g_\alpha \otimes R^{(2)}$. Hence $\sum s(R^{(2)} \leftharpoonup \alpha) R^{(1)} = \sum s(R^{(2)})\, R^{(1)}\, g_\alpha = ug_\alpha$, and part (b) is established.

To show part (c), we first note that $vu = g(g_\alpha)^{-1} = gg_{\alpha^{-1}}$ by part (b) of Proposition 3 and part (b) above. Therefore $h = vu$ and is a grouplike element. By Eq. (5) it follows that $s^4(a) = hah^{-1}$ for all $a \in A$. Thus part (c) follows, and the proof of the theorem is complete.

A finite-dimensional Hopf algebra is said to be unimodular if $\alpha = \varepsilon$, or equivalently left and right integrals for A are the same. Thus by part (a) of Theorem 2 we have:

COROLLARY 1. *Suppose that (A, R) is a finite-dimensional quasitriangular Hopf algebra over the field k. If A is unimodular, then $g = vu$, where g is the distinguished grouplike element of A.*

The Quantum Casimir of a quasitriangular Hopf algebra (A, R) over k is defined [2, p. 66] to be the element $c = us(u)$, where u is described in the preceding theorem. As a corollary to part (c) of Theorem 2 we have:

COROLLARY 2. *Suppose that (A, R) is a finite-dimensional quasitriangular Hopf algebra over the field k. Write $R = \sum R^{(1)} \otimes R^{(2)}$ and let $u = \sum s(R^{(2)})\, R^{(1)}$. Then the Quantum Casimir $c = u^2 l$ for some grouplike element $l \in G(A)$.*

If (A, R) is a finite-dimensional quasitriangular Hopf algebra over the field k, then the product vug_α does not depend on R by part (b) of Theorem 2. The element u may very well depend on R, as an easy example shows.

Suppose that $n > 1$ and that k has a primitive n^{th} root of unity ω. Let $A = k[G]$ be the group algebra of the cyclic group $G = (a)$ of order n over k. Then $(A, 1 \otimes 1)$ is quasitriangular since A is cocommutative. For this particular structure $u = 1$.

Now set

$$R = \frac{1}{n}\left(\sum_{0 \leqslant l, m < n} \omega^{-lm} a^l \otimes a^m\right).$$

Then (A, R) is quasitriangular. To see this, let $\eta \in G(A^*)$ be the algebra homomorphism determined by $\eta(a) = \omega$. Since distinct grouplike elements are linearly independent [6, Proposition 3.2.1.b)], it easily follows that the cyclic group $G' = (\eta)$ has order n, and thus $A^* = k[G']$ is the group algebra of G' over k. Now $f\colon A^* \to A$ defined by $f(\eta^l) = a^l$ is a bialgebra isomorphism. Using the equation $\sum_{l=0}^{n-1} \omega^{ul} = n\delta_{0,u}$ for $u = 0, ..., n-1$ it is easy to see that $f = F(R)$, where $F\colon A \otimes A \to \mathrm{Hom}_k(A^*, A)$ is the identification defined in Section 2. Therefore (QT.1)–(QT.4) are satisfied. Since A is commutative and cocommutative (QT.5) is satisfied also. Thus (A, R) is quasitriangular.

For this particular R it is easy to see that

$$u = \frac{1}{n}\left(\sum_{0 \leqslant m, l < n} \omega^{l(m-l)} a^m\right).$$

If $n = 2$ then $u = a$. Suppose that $n = 3$. Then

$$u = \tfrac{1}{3}\left((1 + 2\omega^2)\, 1 + (2 + \omega)\, a + (2 + \omega)\, a^2\right).$$

In this case note that u and the Quantum Casimir

$$c = \frac{1 - \omega}{3}\,(\omega 1 + a + a^2)$$

are not grouplike elements.

We conclude this paper by finding a necessary and sufficient condition for s^2 to be inner, where s is the antipode of a finite-dimensional Hopf algebra over k, in the spirit of Proposition 1. The proposition below is motivated by the proof of part (a) of Theorem 2.

PROPOSITION 4. *Let A be a finite-dimensional Hopf algebra with antipode s over the field k. Suppose that $\Lambda \in A$ is a non-zero left integral. Then the following are equivalent:*

(a) *s^2 is inner,*

(b) *there are $R, U \in A \otimes A$ such that $\sum \Lambda_{(2)} \otimes \Lambda_{(1)} = \sum R(\Lambda_{(1)} \otimes \Lambda_{(2)})U$.*

Proof. We first show that part (a) implies part (b). Suppose that $u \in A$ is invertible and that $s^2(a) = uau^{-1}$ for all $a \in A$. By Eq. (9) we have $\sum A_{(2)} \otimes A_{(1)} = \sum A_{(1)} \otimes s^2(A_{(2)})\, g$, where g is the distinguished grouplike element of A. Let $R = 1 \otimes u$ and $U = 1 \otimes u^{-1}g$.

We show part (b) implies part (a). Assume that $R, U \in A \otimes A$ satisfy $\sum A_{(2)} \otimes A_{(1)} = \sum R(A_{(1)} \otimes A_{(2)})\, U$. Write $R = \sum R^{(1)} \otimes R^{(2)}$ and $U = \sum U^{(1)} \otimes U^{(2)}$. Then by Lemma 3 we have that

$$\begin{aligned}\sum R(A_{(1)} \otimes R_{(2)})\, U &= \sum R^{(1)} A_{(1)} U^{(1)} \otimes R^{(2)} A_{(2)} U^{(2)} \\ &= \sum A_{(1)} \otimes R^{(2)} s^{-1}(R^{(1)})\, A_{(2)} s(U^{(1)} \leftharpoonup \alpha)\, U^{(2)},\end{aligned}$$

where α is the distinguished grouplike element of A^*. Hence $\sum A_{(2)} \otimes A_{(1)} = \sum A_{(1)} \otimes uA_{(2)}v$ for some $u, v \in A$. But by Eq. (9) again, we have that $\sum A_{(1)} \otimes s^2(A_{(2)})\, g = \sum A_{(1)} \otimes uA_{(2)}v$. This last equation is equivalent to $s^2(A \leftharpoonup p)\, g = u(A \leftharpoonup p)\, v$ for all $p \in A^*$. Since $(A, \leftharpoonup)$ is a free right A^*-module with basis A, this last equation implies that $s^2(a)\, g = uav$ for all $a \in A$. Therefore u is invertible and has inverse vg^{-1}, and hence $s^2(a) = uau^{-1}$ for all $a \in A$. This concludes the proof of the proposition.

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